Vacuum boundary conditions for helicon waves in a cylindrical plasma

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Abstract. We consider helicon propagation in a cylindrical resistive plasma with vacuum boundary conditions. We show that the wave fields (which we assume to be continuous at the boundary) develop rapid variations as $\eta \to 0$, which can be represented physically by surface currents and a surface dipole layer. The contribution of the dipole layer to $\boldsymbol{n} \,.\, [E]$ clears up a difficulty in a recent paper by Francey and Gates in 1968.

In a recent paper (Francey and Gates 1968, to be referred to as FG) the theory of helicon waves in a non-resistive cylindrical plasma with vacuum boundaries is considered. The authors derive a new boundary condition (FG, equation (3.7)), which enables them to solve for the dispersion relation. The same problem has also been investigated by Klozenberg *et al.* (1965, to be referred to as KMT) by solving the equations for finite resistivity η and then taking the limit $\eta \to 0$. Comparing the two approaches (for the $m = 0$ mode), we see three disagreements:

(i) The dispersion relation obtained by KMT includes an imaginary term which gives finite damping due to surface currents.

(ii) The tangential components b_θ, b_z are both discontinuous at the plasma surface in the treatment of KMT; FG obtain the result that b_z is continuous and b_θ discontinuous.

(iii) FG conclude (4th line, p. 713) that the surface currents are zero, which together with the equation

$$\boldsymbol{n} \times [\boldsymbol{b}] = \frac{4\pi}{c} \boldsymbol{j}^* \tag{1}$$

implies that b_θ should be continuous in their solution. In fact, this is not the case, indicating an internal inconsistency in their work.

It is not difficult to resolve these differences, and we shall show how this is done. From FG (equation (3.2)) we have (note that we change all units into Gaussian c.g.s.)

$$\boldsymbol{j} = \frac{1}{\eta(1+\xi^2)} \{\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \,.\, \boldsymbol{E}) + \xi \hat{\boldsymbol{B}} \times \boldsymbol{E} + \xi^2 \boldsymbol{E}\}$$

$$\xi = \frac{\eta nec}{B_0} \tag{2}$$

and we want to take the limit $\eta \to 0$, looking for surface currents. Hence we need the expressions for $\boldsymbol{E}$, and these may be obtained from the field equation (FG, equation (2.3))

$$\boldsymbol{E} = \frac{1}{nec} \boldsymbol{j} \times \boldsymbol{B} + \eta \boldsymbol{j}.$$

Of course, this is the inverse of equation (2), but this indirect treatment turns out to shed some light on the physical phenomena which take place at the boundary. Now KMT show that $\boldsymbol{b}$ (in the plasma) is the sum of two solutions, each of which satisfy the equation $\nabla \times \boldsymbol{b} = \beta \boldsymbol{b}$, where

$$\mathrm{i}\nu\beta^2 - \Omega k \beta + \omega \left(\frac{\Pi^2}{c^2}\right) = 0$$

$$\Pi^2 = \frac{4\pi n e^2}{m}, \qquad \Omega = \frac{eB_0}{mc}. \tag{3}$$

For $\eta \to 0$ one of the two solutions of equation (3) becomes infinite, and it is the corresponding $\boldsymbol{b}$ field which gives the surface effects. Evaluating all components of both $\boldsymbol{E}$ and $\boldsymbol{b}$ for this 'surface' term, we have

$$\begin{aligned}
b_r &\sim 0 \\
b_\theta &\sim -\mathrm{i}\exp\{\Omega\tau k(r-a)\} \\
b_z &\sim \exp\{\Omega\tau k(r-a)\}. \\
E_r &\sim -\frac{a^2\omega_0 k\Omega\tau}{c}\exp\{\Omega\tau k(r-a)\} \\
E_\theta &\sim 0 \\
E_z &\sim -\frac{\mathrm{i}a^2\omega_0 k}{c}\exp\{\Omega\tau k(r-a)\}.
\end{aligned} \tag{4}$$

In the following discussion it is essential to remember that, although the amplitudes of these terms have to be determined by the solution of the boundary value problem, the relative amplitudes of the six quantities are fixed by equation (4). Now KMT show that the amplitude of the discontinuities in b_θ and b_z remain finite as $\eta \to 0$; hence we draw two conclusions about the amplitude of the components of $\boldsymbol{E}$:

(i) The tangential components have a finite discontinuity as $\eta \to 0$.

(ii) E_r becomes a delta function in this limit.

We define the surface current $\boldsymbol{j}^*$ by

$$\boldsymbol{j}^* = \lim_{\eta\to 0,\ \delta\to 0}\int_{a-\delta}^{a} \boldsymbol{j}(r)\,\mathrm{d}r$$

and use equations (2) and (4) to evaluate this quantity. We obtain

$$\begin{aligned}
j_z^* &\sim -\frac{\mathrm{i}c}{4\pi} \\
j_\theta^* &\sim -\frac{c}{4\pi}.
\end{aligned}$$

It is easy to check that equation (1) is satisfied with this surface current and the discontinuities in $\boldsymbol{b}$ given above.

Referring again to FG, we see that the error in their argument comes in the use of equation (3.2). The term $\xi\hat{\boldsymbol{B}}\times\boldsymbol{E}$ cannot be neglected, since E_r becomes infinite at the surface and gives a finite contribution to an integral normal to the surface. It cannot be argued that because E_z is zero inside the plasma it is zero in equation (2), since it is discontinuous at the surface. Physically both of these phenomena can be described by the introduction of a dipole layer at the surface, as suggested by Woods (1962, 1964). From equation (4) the density of this dipole layer is given by

$$\tau \sim -\frac{a^2\omega_0}{4\pi c}$$

and it is easily verified that the relation

$$\boldsymbol{n}\times[\boldsymbol{E}] = 4\pi\boldsymbol{n}\times\nabla\tau$$

is satisfied with the discontinuities of E given in equation (4). Further work is in progress to derive useful boundary conditions for use in the $\eta = 0$ limit.

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The hemispherical box: an example of virtual symmetry

Abstract. An example of accidental degeneracy in quantum mechanics arising from virtual symmetry is exhibited and discussed.

McIntosh (1964) has emphasized that a quantum-mechanical system may inherit a degeneracy through the symmetry of a larger system in which it can be embedded. We wish to report a pleasing example of this afforded by the motion of a single particle in a hemispherical box with impenetrable walls. The method of embedding used was originated by Heisenberg (private communication to Professor C. A. Coulson), though it has apparently not been published by him. Previous applications have dealt mainly with Hückel molecular orbital calculations and the study of lattice vibrations, references to which may be found in the paper by McIntosh. The eigenvalue problem pertaining to the present system, though it is rather simple and could otherwise be solved by straightforward methods, is of interest because of the manner in which it exhibits accidental degeneracy as we shall now describe.

The manifest symmetry group of the hemispherical box is $C_{\infty v}$, which would imply at most a twofold degeneracy in the energy levels. The actual degeneracy is much higher. To show this we follow Heisenberg's procedure as described by McIntosh and attempt to embed in a larger more symmetrical system, in this case the spherical box, so that out of a subset of the eigenfunctions of the latter we may construct the eigenfunctions of the original system. Now, although the extra symmetry of the larger system may imply additional degeneracy, the question still remains as to whether in the subset of eigenfunctions of the smaller system the residual degeneracy is greater than would otherwise have been anticipated. In the present case this does occur. Indeed, since the symmetry group of the spherical box is SO(3, R), we first of all observe that each of its energy levels may be indexed with the angular quantum number l with a $(2l+1)$-fold degeneracy. Then if we select those eigenfunctions which have a node on (say) the xy plane and apply to them the projection operator P defined by

$$(Pf)(z) = \begin{cases} f(z) & z > 0 \\ 0 & z \leqslant 0 \end{cases}$$